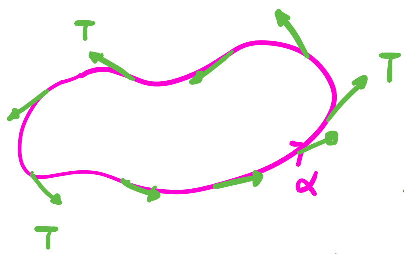
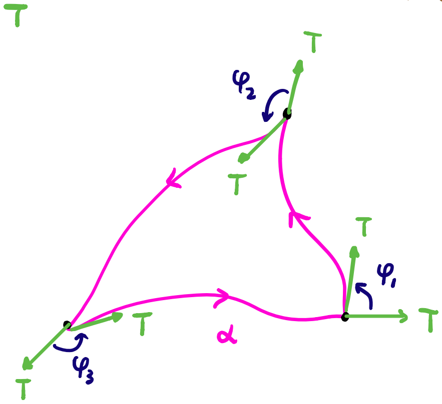
§ The Gauss-Bonnet Theorems

Recall: Theorem of Turning Tangents



$$\int_{\alpha} k(s)\,ds = 2\pi$$

It also works for "piecewise smooth" curves:

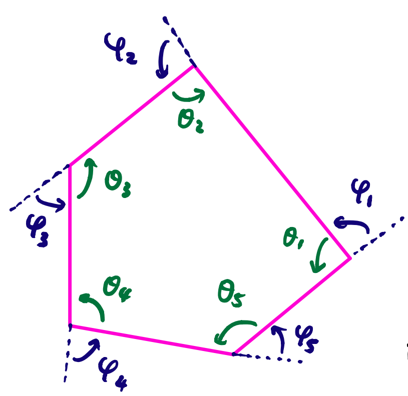


$$\boxed{\int_{\alpha} k(s)\,ds + \sum_i \varphi_i = 2\pi} \quad (*)$$

Special case: When all the smooth pieces are geodesics

(*) $\Rightarrow$ exterior angle sum of polygons $= 2\pi$

$\Updownarrow$



interior angle sum of n-gon $= (n-2)\pi$

i.e. $\sum_{i=1}^{n} \varphi_i = 2\pi$, $\sum_{i=1}^{n} \theta_i = (n-2)\pi$.

Question: What about surfaces? Is the "total curvature" of a surface a constant?

Gauss Bonnet Theorem I:

Let S be a compact orientable surface without boundary.

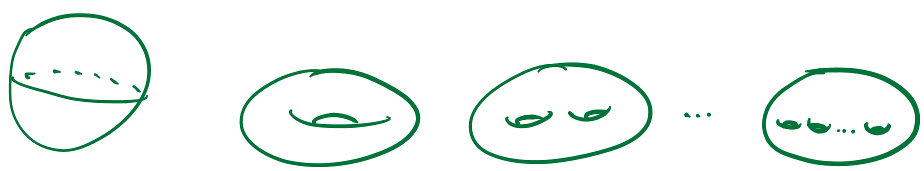
Then

$$\boxed{\int_S K \, da = 2\pi \chi(S)}$$

where $\chi(S)$ is the Euler characteristics of S given by the formula $\chi(S) = 2 - 2g$.

$\llcorner$ genus of S

Examples:



genus:	0	1	2	...	g
χ:	2	0	-2	...	$2-2g$
	>0	$=0$	<0		

Note: The Gauss-Bonnet Theorem is a beautiful result since it relates "geometry" on L.H.S. to "topology" on R.H.S.

There is a version for surfaces with boundary.

Gauss-Bonnet Theorem II:

Let S be a compact orientable surface with boundary.

Then,

$$\boxed{\int_S K \, da + \int_{\partial S} k_g \, ds = 2\pi \chi(S)}$$

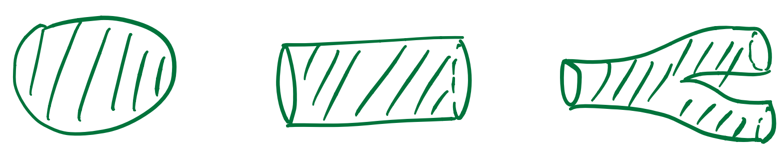
∂S ← positively oriented

where $\chi(S) = 2 - 2g - \gamma$

g = genus

γ = # of boundaries

Example:



g :	0	0	0
γ :	1	2	3
χ :	$\underbrace{1}_{>0}$	$\underbrace{0}_{=0}$	$\underbrace{-1}_{<0}$

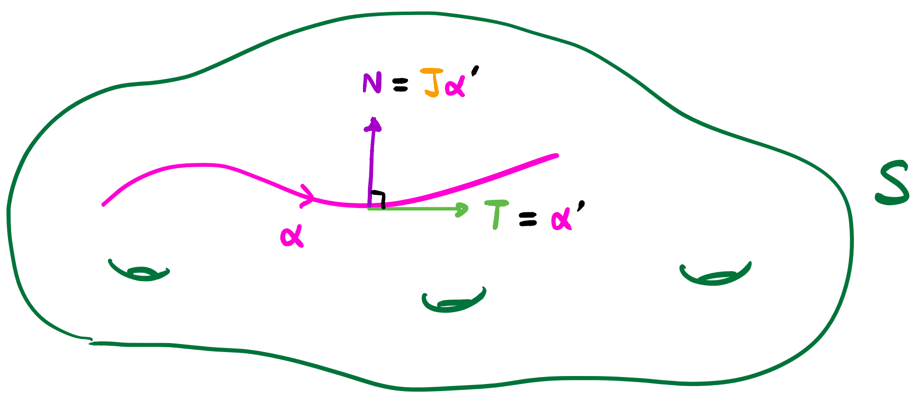
<u>Definition</u>: Let S be an oriented surface, then

$\exists \; J : T_pS \to T_pS$ rotation by $90°$ on each tangent plane.

If $\alpha : I \to S$ is a curve p.b.a.l.,

we define the geodesic curvature of α

$$\boxed{k_g := \langle \nabla_{\alpha'}\alpha' , J\alpha' \rangle}$$



<u>Remark</u>: (1) If S = plane (with standard orientation)

then k_g agrees with the usual curvature k for plane curves.

(2) Switch the orientation of α changes the sign of k_g

(3) α geodesic $\iff k_g \equiv 0$

(i.e. geodesic curvature measures how much is the curve deviated from a geodesic.)

There is also a version of Gauss-Bonnet Theorem which allows the boundary to be only "piecewise smooth":

$$\boxed{\int_S K \, da + \int_{\partial S} k_g \, ds + \underbrace{\sum_i \varphi_i}_{\text{sum of exterior angles.}} = 2\pi \chi(S)}$$

Gauss-Bonnet Theorem III

§ Applications of Gauss-Bonnet Theorems

(I) Any compact orientable surface S without boundary and $K > 0$ everywhere is homeomorphic to a sphere.

Proof: By Gauss-Bonnet I,

$$\int_S K \, da = 2\pi \chi(S)$$

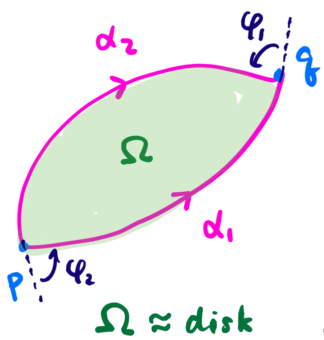
Since $K > 0$ everywhere, the L.H.S. > 0, hence

$$\chi(S) > 0 \implies S \approx \text{sphere}.$$

$\square$

(II) If S is a surface with $K \le 0$ everywhere, then any two geodesics α_1, α_2 starting at the same point $p \in S$ cannot meet again at some $q \in S$ s.t. α_1, α_2 together bounds a "disk".

<u>Proof</u>: By contradiction. Suppose not.



Apply Gauss-Bonnet II to Ω

$$\int_\Omega K\,da + \int_{\partial\Omega} k_g\,ds + \varphi_1 + \varphi_2 = 2\pi\chi(\Omega)$$

Since $\partial\Omega = \alpha_1 \cup \alpha_2$ are geodesics

$\Rightarrow k_g \equiv 0$.

Since $\Omega \approx$ disk

$\Rightarrow \chi(\Omega) = 1$

Therefore,

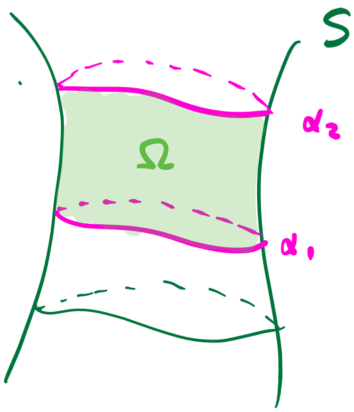
$$\underbrace{\int_\Omega K\,da}_{\le 0} + \underbrace{\varphi_1}_{<\pi} + \underbrace{\varphi_2}_{<\pi} = 2\pi$$

Contradiction!

$\square$

(III) If S is homeomorphic to a cylinder and $K < 0$ everywhere on S
then $\exists$ at most 1 simple closed geodesic on S

<u>Sketch of Proof:</u> Suppose NOT. We have 2 such geodesics α_1, α_2 which together must bound a cylinder in S



Apply Gauss-Bonnet II to Ω

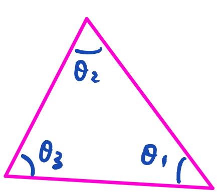
$$\underbrace{\int_\Omega K\,da}_{<0} + \cancel{\int_{\partial\Omega} k_g\,ds}^{\,0} = 2\pi \cancel{\chi(\Omega)}^{\,0}$$

Contradiction!

$\square$

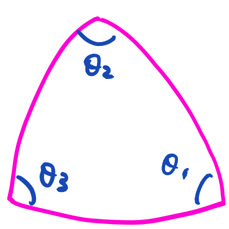
(IV) Interior angle sum of "geodesic triangles"

$K \equiv 0$



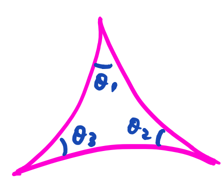
$\sum \theta_i = \pi$

$K > 0$



$\sum \theta_i > \pi$

$K < 0$



$\sum \theta_i < \pi$

§ Proof of Gauss-Bonnet Theorems

We will prove Gauss-Bonnet Theorem I, i.e.

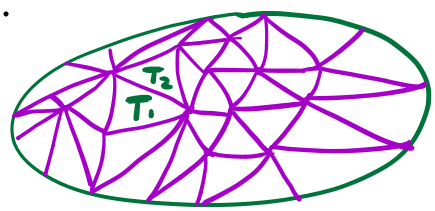
$$\boxed{\int_S K \, da = 2\pi \chi(S)}$$

for any compact orientable surface S without boundary

First, we reduce the problem to the "local" case by

FACT: Any orientable surface can be "triangulated".

e.g.



$$S = \bigcup_i T_i$$

By subdivision we can assume that each T_i is small enough that it is contained in a single coordinate neighborhood.

FACT: Any compact orientable surface can be covered by "conformal coordinate systems":

i.e. $X : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$

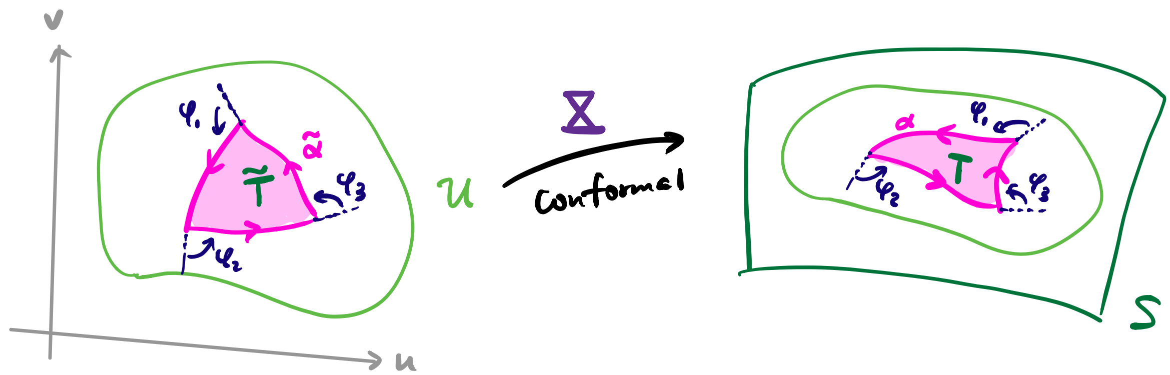
s.t. $\underbrace{(g_{ij})}_{1^{st} \text{ f.f.}} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ for some smooth positive function $\lambda : \mathcal{U} \to \mathbb{R}$.

Note: Conformal coordinates preserve infinitesimal angles!

We will prove the following local Gauss-Bonnet Theorem:

Local Gauss-Bonnet:

Let $\mathbf{X} : \mathcal{U} \subseteq \mathbb{R}^2 \longrightarrow S$ be a conformal parametrization.



$$\boxed{\int_T K \, da + \int_{\partial T} k_g \, ds + \sum_i \varphi_i = 2\pi}$$

Proof: In conformal coordinates,

$$k_g = \frac{1}{2\lambda}\left(\frac{\partial \lambda}{\partial u} v' - \frac{\partial \lambda}{\partial v} u' \right) + k$$

where k = curvature of $\tilde{\alpha}$ (as a plane curve in $\mathbb{R}^2$)

$\tilde{\alpha}(s) = (u(s), v(s))$ p.b.a.l. as a plane curve

$$\int_{\partial T} k_g \, ds + \sum_i \varphi_i = \int_{\tilde{\alpha}} \frac{1}{2\lambda}\left(\frac{\partial \lambda}{\partial u} v' - \frac{\partial \lambda}{\partial v} u' \right) + \underbrace{\int_{\tilde{\alpha}} k + \sum_i \varphi_i}_{= \, 2\pi \text{ (Thm. of Turning tangent)}}$$

By Green's Theorem,

$$\int_{\tilde{\alpha}} \frac{1}{2\lambda^2}\left(\frac{\partial \lambda^2}{\partial u} v' - \frac{\partial \lambda^2}{\partial v} u' \right) = \iint_{\tilde{T}} \frac{\partial}{\partial u}\left(\frac{1}{2\lambda} \frac{\partial \lambda}{\partial u} \right) + \frac{\partial}{\partial v}\left(\frac{1}{2\lambda} \frac{\partial \lambda}{\partial v} \right)$$

$$= \iint_{\tilde{T}} \frac{1}{2} \Delta (\log \lambda) \, du\, dv$$

$$= \iint_{\tilde{T}} \underbrace{\frac{1}{2\lambda} \Delta (\log \lambda)}_{-K} \underbrace{\lambda \, du\, dv}_{da}$$

$$= -\int_{T} K \, da$$

We now apply local Gauss-Bonnet to each T_i in a "fine" triangulation with

$$\begin{cases} F = \#\text{ faces} \\ E = \#\text{ edges} \\ V = \#\text{ vertices} \end{cases}$$

Recall:

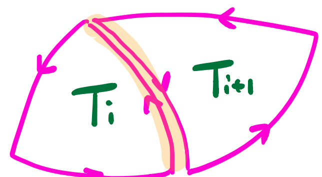
$$\chi(S) = F - E + V$$

$$\int_{T_i} K\, da + \int_{\partial T_i} k_g\, ds + \sum_j \varphi_j^i = 2\pi$$

Sum over all the T_i's, we get

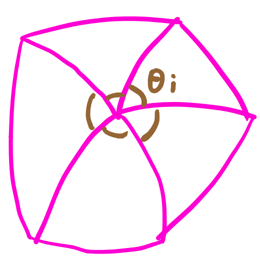
$$\int_S K\, da + 0 + \begin{matrix} 3\pi F \\ -2\pi V \end{matrix} = 2\pi F$$

Reason: (I)



different orientations $\Rightarrow$ cancels

(II)



ext. ∠ sum = 3π − int. ∠ sum

$\Downarrow$

$$\sum_{i,j} \varphi_j^i = 3\pi F - 2\pi V$$

Hence, $$\int_S K da = 2\pi F - \underbrace{3\pi F}_{= 2\pi E} + 2\pi V$$

since $3F = 2E$

$\square$